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Algorithms for Determination of Period-Doubling Bifurcation Points in Ordinary Differential Equations

MILAN KUBÍČEK AND MARTIN HOLODNIOK

Department of Mathematics and Computer Center, Prague Institute of Chemical Technology, 166 28 Praha 6, Czechoslovakia

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Four algorithms for evaluating period-doubling bifurcation points in periodic solutions of autonomous systems of ordinary differential equations are presented. Two of the algorithms can also be used for nonautonomous systems. The algorithms are applied to three examples: two interconnected reaction cells, the Lorenz model and a reactor with periodic forcing. Convergence properties of the algorithms are shown and a number of computed bifurcation points are presented. Convergence of the Feigenbaum sequence is shown for a period-doubling cascade in the Lorenz model. © 1987 Academic Press, Inc.

1. INTRODUCTION

Numerical methods in the bifurcation theory of nonlinear dynamic systems are currently the subject of much attention. Methods for investigating branching of stationary solutions (including Hopf's bifurcations) are found in [1–4].

A continuation algorithm for construction of the dependence of periodic solutions on a parameter is described in [5]. The branching of periodic solutions is reviewed by Sattinger [6] and discussed in [7]. Numerical methods for determination of branching points of periodic solutions are studied by Becker and Seydel [8] in the case of the limit point and by the present authors [9] in the case of the point of period-doubling bifurcation.

Several authors have evaluated the cascade of period-doubling bifurcations on the stable branch by means of sequentially applied dynamic simulation [10–12] in order to verify Feigenbaum's theory [13–15] in connection with the transition to chaotic behaviour of the system.

2. DEVELOPMENT OF ALGORITHMS FOR AUTONOMOUS SYSTEMS

Consider an autonomous system of ordinary differential equations

$$\frac{dy_i}{dt} = f_i(y_1, ..., y_n, \alpha), \qquad i = 1, 2, ..., n, \tag{1}$$

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depending on a parameter α. A periodic solution with period T satisfies

$$y_i(t+T)=y_i(t), \qquad i=1, 2,..., n. \tag{2}$$

Substituting $t=Tz$ in (1) generates a system of equations

$$\frac{dy_i}{dz}=Tf_i(y_1,..., y_n, \alpha), \qquad i=1, 2,..., n, \tag{3}$$

with mixed boundary conditions in the form

$$y_i(1)-y_i(0)=0, \qquad i=1, 2,..., n. \tag{4}$$

For the shooting method choose initial conditions

$$y_i(0)=x_i, \qquad i=1, 2,..., n, \tag{5}$$

as well as values of the period T and the parameter α. Then system (3) is integrated from $z=0$ to $z=1$. Values of the solution at $z=1$ are expressed as

$$y_i(1)=\varphi_i(x_1,..., x_n, T, \alpha), \qquad i=1, 2,..., n. \tag{6}$$

Relation (4) has to be valid for any periodic solution. Hence, we have to satisfy n equations

$$F_i(x_1,..., x_n, T, \alpha)=\varphi_i(x_1,..., x_n, T, \alpha)-x_i=0, \qquad i=1, 2,..., n, \tag{7}$$

with $n+1$ unknowns $x_1,..., x_n$, T, and one parameter α. To obtain a periodic solution for a fixed α assign a value to x_k for some k [5]. The period T cannot be assigned because the solution of Eqs. (7) exists for discrete (and a priori unknown) values of T only. The procedure will be successful if the chosen value actually exists on the trajectory of the kth component of the desired periodic solution $y_k(z)$, $z\in[0, 1)$.

Stability of the computed periodic solution is determined on the basis of Floquet (characteristic) multipliers, (see [7]), which are the eigenvalues λ of the monodromy matrix

$$B=\{\partial\varphi_i/\partial x_j\}. \tag{8}$$

Elements of the monodromy matrix (and values of $\partial F_i/\partial x_j$ in (7)) are evaluated on the basis of variational differential equations for variational variables

$$p_{ij}(z)=\partial y_i/\partial x_j, \qquad i, j=1,..., n. \tag{9}$$

These differential equations obtained by differentiating Eqs. (3) with respect to x_j, are in the form

$$\frac{dp_{ij}}{dz}=T\sum_{l=1}^{n}\frac{\partial f_i}{\partial y_l}p_{lj}, \qquad p_{ij}(0)=\delta_{ij} \tag{10}$$

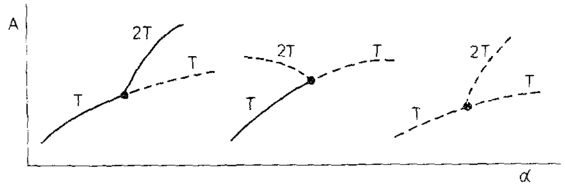

FIG. 1. Schematic representation of bifurcation at the point of period-doubling. (——) stable periodic solutions, (- - -) unstable periodic solutions. A is the amplitude or other representative value of the periodic solution.

(δ_{ij} is the Kronecker delta). For elements of the monodromy matrix we have

$$B = \{p_{ij}(1)\}. \tag{11}$$

Consider a branch of periodic solutions which depend on the value of parameter α. Stability of the periodic solutions may change at the bifurcation value of parameter α where a certain characteristic multiplier of the corresponding periodic solution lies on the unit circle. This multiplier will be either $+1$, or -1, or imaginary. The first case corresponds to limit (turning) points or bifurcation (crossection, symmetry breaking) points on dependence curves of periodic solutions on a parameter. The third case mostly indicates a bifurcation to an invariant torus. Here we shall deal with the second case, that is, with the period-doubling bifurcation points. At this point (where the characteristic multiplier passes through -1) a new branch of periodic solutions bifurcates with the period which has approximately (asymptotically) doubled in comparison with the period on the original branch. The situation is schematically shown on Fig. 1; more detailed explanation in [7].

The goal of this paper is to construct computational algorithms for direct (iterative) determination of period-doubling bifurcation points. We will determine a periodic solution whose characteristic multiplier is -1. In the following we describe four iterative algorithms constructed for this purpose.

Algorithm I

Let the characteristic polynomial of the monodromy matrix B be $P(\lambda) = (-1)^n \det(B - \lambda I)$, i.e.,

$$P(\lambda) = \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n. \tag{12}$$

The coefficients a_j are evaluated by the Krylov method [16]. $\lambda = -1$ is the root of the characteristic polynomial (12) if

$$F_{n+1}(x_1, ..., x_n, T, \alpha) = 1 + \sum_{i=1}^{n} (-1)^i a_i = 0. \tag{13}$$

As a result we obtain $n+1$ nonlinear (algebraic) equations (7) and (13) for $n+1$ unknowns $x_1, ..., x_{k-1}, x_{k+1}, ..., x_n, T, \alpha$. The Newton method is used to solve this system. The first n rows of the Jacobi matrix are evaluated on the basis of the variational variables,

$$\frac{\partial F_i}{\partial x_j} = p_{ij}(1) - \delta_{ij}, \qquad \frac{\partial F_i}{\partial T} = f_i(y(1), \alpha),$$
$$\frac{\partial F_i}{\partial \alpha} = q_i(1), \qquad i = 1, 2, \ldots, n, \tag{14}$$

where the variational variables p_{ij} and $q_i = \partial y_i / \partial \alpha$ satisfy variational equations (10) and

$$\frac{dq_i}{dz} = T\left[\sum_{l=1}^{n} \frac{\partial f_i}{\partial y_l} q_l + \frac{\partial f_i}{\partial \alpha}\right], \qquad q_i(0) = 0, \qquad i = 1, 2, \ldots, n. \tag{15}$$

The elements of the last row of the Jacobi matrix $\partial F_{n+1}/\partial x_j$, $\partial F_{n+1}/\partial T$, $\partial F_{n+1}/\partial \alpha$ are evaluated by means of difference formulas. An "analytical" method using variational variables is also possible. However, it is very cumbersome for large n.

The proposed algorithm:

(1) Choose initial estimates of $x_1, \ldots, x_n$, T, α. The value of x_k remains fixed in the iteration process.

(2) Integrate the set of $n(n+2)$ differential equations (3), (10), (15) with the corresponding initial conditions from $z = 0$ to $z = 1$. Evaluate residuals $F_1, \ldots, F_{n+1}$ according to (7) and (13). Evaluate the first n rows of the Jacobi matrix according to Eqs. (14).

(3) Evaluate the last row of the Jacobi matrix by means of finite differences. The above mentioned set of $n(n+2)$ differential equations must be integrated $n+1$ times to obtain the finite difference approximations. Equations (15) are not integrated in this step.

(4) Compute next Newton iteration. If the prescribed accuracy is not fulfilled, go to step (2).

Algorithm II

The characteristic polynomial (12) must have one root equal to unity because we consider the periodic solution of the autonomous system (3) (e.g., [7, 17]). The polynomial (12) can be decomposed into the form

$$P(\lambda) = (\lambda + 1)(\lambda - 1)(\lambda^{n-2} + p_1 \lambda^{n-3} + \cdots + p_{n-3}\lambda + p_{n-2}) + C\lambda + D, \tag{16}$$

where the coefficients $p_1, \ldots, p_{n-2}$, C, D are evaluated recursively:

$$p_1 = a_1; \qquad p_2 = a_2 + 1; \qquad p_k = a_k + p_{k-2}, \qquad k = 3, 4, \ldots, n-2;$$
$$C = a_{n-1} + p_{n-3}, \qquad D = a_n + p_{n-2}. \tag{17}$$

Determining the periodic solution, given that $C=0$ and $D=0$, produces a period-doubling bifurcation point. Therefore, two additional equations

$$F_{n+1}(x_1, ..., x_n, T, \alpha) = C = 0 \tag{18}$$

and

$$F_{n+2}(x_1, ..., x_n, T, \alpha) = D = 0 \tag{19}$$

are to be satisfied at such a bifurcation point. As a result we obtain $n+2$ nonlinear equations for $n+1$ unknowns. Use the Gauss–Newton iteration method (see, e.g., [18]) to solve this system. The Gauss–Newton method seeks the minimum of the function

$$\phi(x_1, ..., x_n, T, \alpha) = \sum_{i=1}^{n+2} F_i^2(x_1, ..., x_n, T, \alpha). \tag{20}$$

In case the obtained minimum is zero, the results correspond to a period-doubling bifurcation point. The Jacobi matrix of the system is needed for application of the Gauss–Newton method analogously to Algorithm I. Here we compute the two last rows of the $(n+2) \times (n+1)$ Jacobi matrix by means of finite differences. The algorithm is quite analogous to Algorithm I. Now evaluate $n+2$ residuals (7), (18), (19) instead of $n+1$ residuals as in Algorithm I. Note that F_{n+1} in Eq. (13) is equal to $D-C$ or $C-D$ for n even or odd, respectively.

Algorithm III

The algorithm is based on the same decomposition (16) as Algorithm II. Recurrence relations (17) are also valid. Instead of two equations (18), (19) we add only one equation

$$F_{n+1}(x_1, ..., x_n, T, \alpha) = D = 0. \tag{21}$$

As a result we have a system of $n+1$ nonlinear equations (7), (21) for $n+1$ unknowns. The Newton method is used as in Algorithm I. If a solution of Eqs. (7), (21) is found, it is a periodic solution of (3), $P(\lambda)$ has $+1$ as a root and, therefore, the left-hand side in Eq. (16) is divisible by $(\lambda - 1)$. The first term on the right-hand side of (16) is also divisible by $(\lambda - 1)$. Therefore $C\lambda + D$ must also be divisible by $(\lambda - 1)$. From the validity $D=0$ it follows that $C=0$. Numerical realization of the Newton method is quite analogous to Algorithm I. We can use Eq. (18) instead of Eq. (21) and thus form a modified Algorithm III.

Algorithm IV

The monodromy matrix B has -1 as an eigenvalue at the period-doubling bifurcation point, i.e., there exists a nonzero vector $v = (v_1, ..., v_n)$ such that

$$(B + I)v = 0. \tag{22}$$

	$x_1,\dots,x_{k-1},x_{k+1},\dots,x_n$	T	α	$v_1,\dots,v_{s-1},v_{s+1},\dots,v_n$
Eqs. (7)	X*	X	X	O
Eqs. (22)	X	X	X	X**

FIG. 2. The occurrence matrix for the $2n \times 2n$ system (7), (22) in Algorithm IV. (*) matrix $B-I$ without kth column, (**) matrix $B+I$ without sth column.

Each nonzero multiple of the vector v is also a solution of Eq. (22). We can, therefore, permanently fix one component of the vector v, e.g.,

$$v_s = 1, \qquad s \in [1, n]. \tag{23}$$

As a result we obtain $2n$ equations (7), (22) for $2n$ unknowns $x_1,\dots, x_{k-1}, x_{k+1},\dots, x_n, T, \alpha, v_1,\dots, v_{s-1}, v_{s+1},\dots, v_n$. The Newton method is used to solve this $2n \times 2n$ system. The occurrence matrix (the structure of the Jacobi matrix) is shown in Fig. 2, where the elements, which can be easily evaluated by using the variational variables, are denoted by a circle.

3. ALGORITHMS FOR NONAUTONOMOUS SYSTEMS

The system (1) presented above is autonomous. The situation in nonautonomous systems is very similar. Consider a system

$$\frac{dy_i}{dt} = f_i(t, y_1,\dots, y_n, \alpha), \qquad i = 1, 2,\dots, n, \tag{24}$$

where functions f_i are periodic in time t with a known period T. Then periodic solutions of Eqs. (24) with only periods mT, m a positive integer, can exist. Thus, we have

$$y_i(mT) - y_i(0) = 0, \qquad i = 1, 2,\dots, n, \tag{25}$$

instead of Eq. (4). After choosing the initial conditions (5) and integrating from $t = 0$ to $t = mT$ we obtain, similarly to Eq. (6),

$$y_i(mT) = \varphi_i(x_1,\dots, x_n, \alpha), \qquad i = 1, 2,\dots, n. \tag{26}$$

Equations

$$F_i(x_1,\dots, x_n, \alpha) = \varphi_i(x_1,\dots, x_n, \alpha) - x_i = 0, \qquad i = 1, 2,\dots, n \tag{27}$$

have to be satisfied for the periodic solution with the period mT. The stability of the periodic solution is governed by eigenvalues of the matrix B in (8), the elements of which can be evaluated by using variational equations similar to Eqs. (10). The only difference is that $\lambda = 1$ need not be the eigenvalue of B for nonautonomous systems (more precisely: B has 1 as eigenvalue at bifurcation points, e.g., limit points). From this point of view it is clear that only Algorithms I and IV can be used for evaluation of period-doubling bifurcation points in nonautonomous systems. A branch of periodic solutions with the period $2mT$ then branches off at such points.

4. Applications

We shall demonstrate the effectiveness of the algorithms on three examples.

EXAMPLE 1. Consider two interconnected well-mixed cells where chemical reactions take place. The Brusselator model chemical reaction scheme has been chosen [19].

The governing equations have the following form [20], $n = 4$:

$$\begin{aligned}
\frac{dy_1}{dt} &= A - (B+1)\,y_1 + y_1^2 y_2 + \alpha(y_3 - y_1),\\
\frac{dy_2}{dt} &= By_1 - y_1^2 y_2 + \frac{\alpha}{\rho}(y_4 - y_2),\\
\frac{dy_3}{dt} &= A - (B+1)\,y_3 + y_3^2 y_4 + \alpha(y_1 - y_3),\\
\frac{dy_4}{dt} &= By_3 - y_3^2 y_4 + \frac{\alpha}{\rho}(y_2 - y_4).
\end{aligned} \tag{28}$$

Here A, B, ρ, and α are parameters of the problem, the values $A = 2$, $B = 5.9$, $\rho = 0.1$ are used in computations, α is considered as the bifurcation parameter (it characterizes mass transfer coefficient).

EXAMPLE 2. The Lorenz model [21]. The governing equations are in the form

$$\begin{aligned}
\frac{dy_1}{dt} &= \sigma y_2 - \sigma y_1,\\
\frac{dy_2}{dt} &= r y_1 - y_1 y_3 - y_2,\\
\frac{dy_3}{dt} &= y_1 y_2 - b y_3.
\end{aligned} \tag{29}$$

The Rayleigh number r is considered as the bifurcation parameter α, the remaining parameters are set $\sigma = 16$, $b = 4$ (see, e.g., [22, 23]).

TABLE I

Examples of Convergence of Algorithms I–III, Example 1, i.e., Model (28)

Algorithm	Iteration	x_2	x_3	x_4	T	α	$\|F\|^2$
I	0	4.2	0.89	4.4	4.05	1.17	$7E$-3
	1	4.19770	0.88832	4.44148	4.05079	1.17189	$2E$-5
	2	4.19845	0.88835	4.44238	4.05146	1.17201	$6E$-11
	3	4.19846	0.88835	4.44239	4.05146	1.17201	$3E$-19
II	0	4.2	0.89	4.4	4.05	1.17	$1E$-2[a]
	1	4.19676	0.88849	4.44084	4.05076	1.17186	$3E$-5
	2	4.19845	0.88835	4.44238	4.05146	1.17201	$1E$-10
	3	4.19846	0.88835	4.44239	4.05146	1.17201	$7E$-19
III	0	4.2	0.89	4.4	4.05	1.17	$1E$-2
	1	4.19856	0.88848	4.44238	4.05169	1.17201	$1E$-5
	2	4.19846	0.88835	4.44239	4.05146	1.17201	$1E$-11
	3	4.19846	0.88835	4.44239	4.05146	1.17201	$6E$-22

[a] Equal to ϕ in Eq. (20).

Note. Values $k=1$ and $x_k=2$ have been fixed. Initial guess taken from the results of continuation of periodic solutions. Point 6 (cf. Table III and Fig. 4) is obtained by all three algorithms.

EXAMPLE 3. The third model is nonautonomous. A well-mixed reactor with the Brusselator chemical reaction and external periodic forcing is described by the system of two differential equations [24, 25],

$$\begin{aligned} \frac{dy_1}{dt} &= y_1^2 y_2 - (B+1)\, y_1 + A + \alpha \sin \omega t, \\ \frac{dy_2}{dt} &= By_1 - y_1^2 y_2. \end{aligned} \tag{30}$$

Here α is the bifurcation parameter (amplitude of external forcing) and $A=2$, $B=6$, $\omega=3$ are chosen parameter values. Evidently the period is $T=2\pi/\omega=2.094395$.

All computations are in single precision arithmetic ($\sim$14 decimal digits) on the computer CYBER 175. Examples of the course of the Newton method (Algorithms I–III) for the problem (28) are shown in Table I, for Algorithm IV in Table II. Initial guesses originated approximately from the results of continuation of periodic solutions in dependence on the parameter α. Results of one such continuation [26] (obtained by the DERPER algorithm described in [5]) are presented in Fig. 3. Four period-doubling bifurcation points exist on the isolated and closed dependence curve of periodic solutions on the parameter α. They are presented in Table III as points 1–4. Every bifurcation points is presented four times because the course of $y_1(z)$, $z \in [0, 1)$, intersects four times the line $y_1(z)=x_1=2$ given by the

TABLE II

Examples of Convergence of Algorithm IV

Iteration	x_2	x_3	x_4	T	α	v_2	v_3	v_4	$\|F\|^2$
0	4.2	0.89	4.4	4.05	1.17	1.0	1.0	1.0	$2E2$
1	4.21694	0.89195	4.46158	4.07078	1.17449	−1.23908	0.07552	−0.99297	$5E$-2
2	4.19850	0.88833	4.44245	4.05150	1.17202	−1.30363	0.10569	−1.08017	$9E$-6
3	4.19846	0.88835	4.44239	4.05146	1.17201	−1.30315	0.10582	−1.07959	$1E$-12
4	4.19846	0.88835	4.44239	4.05146	1.17201	−1.30315	0.10582	−1.07959	$2E$-20
0	4.2	0.89	4.4	4.05	1.17	−2.0	0.0	−1.0	$2E0$
1	4.20365	0.88944	4.44770	4.05697	1.17269	−1.30977	0.10875	−1.08891	$1E$-3
2	4.19848	0.88835	4.44241	4.05149	1.17201	−1.30314	0.10582	−1.07960	$5E$-8
3	4.19846	0.88835	4.44239	4.05146	1.17201	−1.30315	0.10582	−1.07959	$4E$-16
4	4.19846	0.88835	4.44239	4.05146	1.17201	−1.30315	0.10582	−1.07959	$2E$-22

Note. Example 1, i.e., Model (28). Values $k=1$, $x_k=2$, $s=1$, and $v_s=1$ have been fixed. Initial guess for x_2, x_3, x_4, T, and α taken from the results of continuation of periodic solutions. Initial guess for v_2, v_3, v_4 has been chosen as 1.0. Point 6 (cf. Table III and Fig. 4) is obtained.

choice $x_k = 2$, $k = 1$, for every periodic solution corresponding to points 1–4 in Table III. Some additional bifurcation points are presented in Table III (cf. the solution diagram in Fig. 4 [26]).

Several period-doubling bifurcation points of the Lorenz model (29) are presen-

TABLE III

Period-Doubling Bifurcation Points of Problem (28)

Point No.		x_2	x_3	x_4	T	α
1	a	4.56568	1.01370	4.80562	12.62766	1.27471
	b	3.02534	0.90140	3.14959		
	c	5.58827	1.61580	5.72439		
	d	3.06067	0.89496	3.18858		
2	a	3.12258	0.86612	3.26192	11.59421	1.22382
	b	3.68848	0.87736	3.88267		
	c	3.01182	0.87130	3.13921		
	d	5.72536	1.69797	5.83894		
3	a	3.25999	0.86603	3.41344	11.47081	1.22556
	b	3.33969	0.86674	3.50115		
	c	3.01846	0.87105	3.14640		
	d	5.64508	1.60500	5.78922		
4	a	3.13884	0.85654	3.28176	11.83646	1.20614
	b	3.60126	0.86378	3.79038		
	c	3.02023	0.86115	3.15017		
	d	5.57275	1.48779	5.75492		
5		2.89396	1.34039	2.98738	13.60085	1.25089
6		4.19846	0.88835	4.44239	4.05146	1.17201
7		3.02995	0.85167	3.16255	8.52194	1.18940
8		3.23090	0.85069	3.38517	17.03632	1.19239
9		5.15225	1.13527	5.41485	34.07662	1.19307
10		3.00563	0.88225	3.13062	8.54342	1.24307
11		3.03094	0.90653	3.15410	8.23751	1.29353
12		5.56151	1.61440	5.69636	16.47603	1.29325
13		3.02443	1.08913	3.12963	4.89756	1.47021
14		5.20789	1.56382	5.33925	9.79634	1.46909
15		5.21049	1.56643	5.34122	19.59387	1.46882
16		4.49352	1.00470	4.73403	18.01466	1.24686
17		4.02058	0.88784	4.24570	20.87305	1.20131
18		5.56447	1.47919	5.74906	23.67249	1.20668

Note. Values $k = 1$ and $x_k = 2$ have been fixed. The numbering of the points is the same as in Figs. 3 and 4. Points 1–4 are on an isolated branch of periodic solutions, see Fig. 3; each point is presented in four different forms (a–d) to demonstrate that the course of $y_1(z)$, $z \in [0, 1)$, has four intersections with $x_1 = 2$. On the other hand, only one representation of points 5–18 (cf. Fig. 4) is presented.

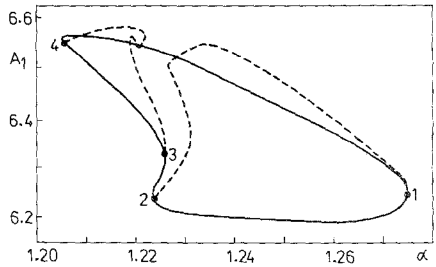

FIG. 3. The solution diagram of periodic solutions of (28) in dependence on the parameter α. A_1 is the amplitude of y_1. Points of period-doubling bifurcation are denoted by (•), the numbers agree with Table III. (——) isolated dependence of periodic solutions with the period $T \approx 11$–13. (- - -) branches of periodic solutions with the double period $T \approx 22$–26.

ted in Table IV. These points have been successfully computed by the aid of all four algorithms. Results in the table correspond to a cascade of period-doubling bifurcations (cf. Fig. 5 [23]). The values of the parameter r at the individual bifurcation points form a Feigenbaum sequence $\{r_j\}$ [13]. The values

$$\delta_j = \frac{r_j - r_{j-1}}{r_{j+1} - r_j} \tag{31}$$

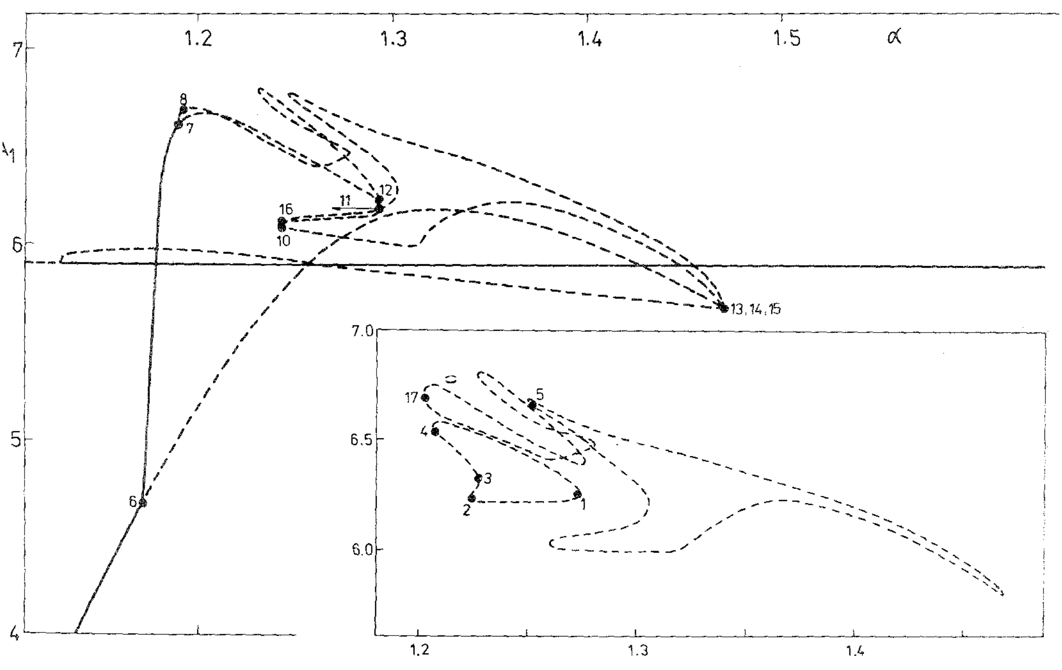

FIG 4. The solution diagram of periodic solutions of (28). A_1 is the amplitude of y_1. Points of period-doubling bifurcation are denoted by (•) and numbered according to Table III. (——) stable, (- - -) unstable.

TABLE IV

A Cascade of Period-Doubling Bifurcation Points in the Lorenz Model (29)

j	$(x_2)_j$	$(x_3)_j$	T_j	r_j	δ_j
1	20.90946	273.34849	0.30618	356.93391	
2	16.85987	246.64055	0.63009	338.06197	4.9740
3	21.19530	259.36006	1.26750	334.26789	4.7313
4	17.29002	244.99724	2.53818	333.46599	4.6824
5	17.24223	244.70901	5.07771	333.29472	4.6707
6	17.25889	244.74356	10.15599	333.25806	

Note. Values $k=1$ and $x_k=3.82038$ have been fixed, r is considered as the bifurcation parameter α, r_j is the bifurcation value on the jth branch of the cascade (cf. Fig. 5). δ_j is defined by Eq. (31).

are presented in Table IV, too. We observe very good convergence to a limit, which is approximately $\delta^* \approx 4.6692$ [13].

A success of individual algorithms in dependence on initial guesses for the model (28), i.e., the Brusselator model, is studied in [27]. Initial guesses are generated randomly from given intervals and all four algorithms are compared for the same

TABLE V

Period-Doubling Bifurcation Points in the Nonautonomous Problem (30), $A=2$, $B=6$, $\omega=3$

m	Point No.	x_1	x_2	α
1	1	1.71831	2.82302	0.85929
	2	1.07223	3.19165	1.67532
2	3	0.90362	4.84987	0.49000
	4	1.19253	5.43587	1.08628
4	5	0.94915	4.71126	0.49267
	6	0.57548	4.18641	0.79980
	7	0.95978	3.59215	0.87795
	8	0.86297	3.54160	1.08437
6	9	0.95481	4.71477	0.49518
	10	2.35876	2.86705	0.69611
	11	1.93724	2.75792	0.81108
	12	0.79190	6.60985	0.81895
	13	0.86362	6.04998	0.91460
	14	4.79358	1.28787	0.92512
	15	3.19709	3.19230	0.97113
	16	1.28865	5.42704	1.08228
7	17	0.74489	4.64816	0.44548
	18	1.36023	3.11399	0.67755

Note. The points are numbered in agreement with Fig. 6.

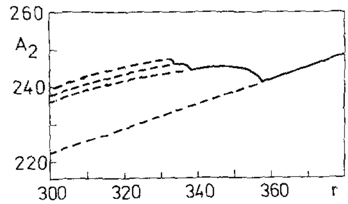

FIG. 5. A cascade of period-doubling bifurcations in the Lorenz model (29) and r is considered as the bifurcation parameter α. For numerical values see Table IV. A_2 is the amplitude of y_2, (——) stable, (– – –) unstable.

guesses (a random choice of v_i is used for Algorithm IV). Generally speaking, all four algorithms are comparable. However, we can obtain different solutions by using different algorithms, because a large number of solutions exist in the model (28) (cf. Table III). Algorithm IV can sometimes be less successful because of a bad guess of v_i (we have no information about their values in advance).

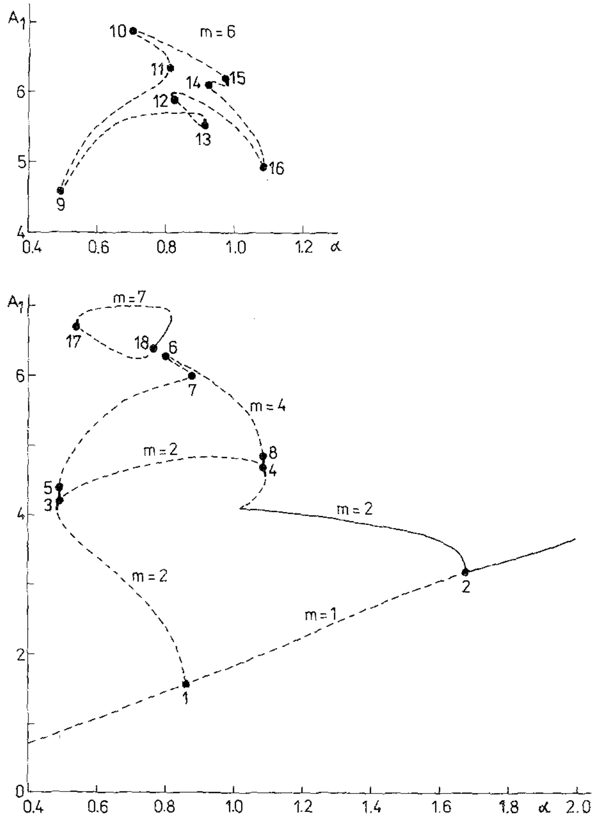

FIG. 6. The solution diagram of m-periodic solutions of the nonautonomous system (30). $A=2$, $B=6$, $\omega=3$ ($T=2.094395$). Points of period-doubling bifurcation are denoted by (•) and numbered in agreement with Table V. (——) stable, (– – –) unstable. (Erratum: $\alpha_{\text{correct}}=\alpha-0.1$ for $m=7$ curve.)

Some resulting period-doubling bifurcation points of the nonautonomous problem (30) are presented in Table V. The solution diagram of periodic solutions is shown in Fig. 6 [28], the period-doubling bifurcation points (on the branches with $m = 1, 2, 4, 6, 7$) are denoted in the figure. We can obtain the solutions for $m = 2$ twice because of a T-shift (generally m-times) if we plot x_1 in dependence on the values of the parameter.

Note that the presented algorithms compute bifurcation values of the parameter to very high accuracy (depending on the accuracy of the integration routine used and the round-off errors of the computer).

5. Conclusions

Four algorithms for evaluating period-doubling bifurcation points presented in this paper can be easily used for most autonomous nonlinear dynamic system of low order, say $n < 20$. The use of the algorithms is limited by the applicability of the shooting method. If the initial value problems are unstable, i.e., there are multipliers of the order 10^5 or higher, the integration, and thus the simple shooting method, usually fails. Multiple shooting methods could be used in such cases. A simple modification of Algorithms I and IV presented in the paper can be used when we have a nonautonomous system with periodic right-hand sides (cf. Example 3). Convergence properties of the algorithms are good, moreover, results from a continuation algorithm can be used as good initial guesses for the Newton method. Starting points on emanating branches of solutions used for the continuation algorithm can be determined [29] after evaluation of the period-doubling bifurcation point.

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References

1. H. D. Mittelmann and H. Weber, "Numerical methods for bifurcation problems - a survey and classification," *Bifurcation Problems and Their Numerical Solution*, edited by H. D. Mittelmann and H. Weber (Birkhäuser-Verlag, Basel, 1980).
2. M. Kubíček and M. Marek, *Computational Methods in Bifurcation Theory and Dissipative Structures* (Springer-Verlag, New York, 1983).
3. T. Küpper, H. D. Mittelmann and H. Weber, Eds., *Numerical Methods for Bifurcation Problems*, ISNM 70 (Birkhäuser-Verlag, Basel, 1984).
4. A. D. Jepson and H. B. Keller, *Numerical Methods for Bifurcation Problems*, ISNM 70 (Birkhäuser-Verlag, Basel, 1984), p. 219.
5. M. Holodniok and M. Kubíček, *J. Comput. Phys.* **55** (1984), 254.

6. D. H. SATTINGER, *Bull. Amer. Math. Soc.* **3** (1980), 779.
7. G. IOOSS AND D. D. JOSEPH, *Elementary Stability and Bifurcation Theory* (Springer-Verlag, New York, 1981).
8. K. H. BECKER AND R. SEYDEL, "A Duffing Equation with More than 20 Branch Points," *in Lecture Notes in Math.* Vol. 878 (Springer-Verlag, New York, 1981), p. 99.
9. M. KUBIČEK AND M. HOLODNIOK, *Numerical Methods for Bifurcation Problems,* ISNM 70 (Birkhäuser-Verlag, Basel, 1984), p. 247.
10. V. FRANCESCINI, *J. Stat. Phys.* **22** (1980), 397.
11. V. FRANCESCINI AND C. TEBALDI, *J. Stat. Phys.* **21** (1979), 707.
12. L. TADESCHINI-LALLI, *J. Stat. Phys.* **27** (1982), 365.
13. M. J. FEIGENBAUM, *J. Stat. Phys.* **6** (1979), 669.
14. M. J. FEIGENBAUM, *Phys. Lett. A* **74** (1979), 375.
15. P. COLLET, J. P. ECKMANN AND H. KOCH, *J. Stat. Phys.* **25** (1981), 1.
16. J. H. WILKINSON AND C. REINSCH, *Handbook for Automatic Computation, II. Linear Algebra* (Springer-Verlag, Berlin, 1971).
17. E. A. CODDINGTON AND N. LEVINSON, *Theory of Ordinary Differential Equations* (McGraw-Hill, New York, 1955).
18. J. M. ORTEGA AND W. C. RHEINBOLDT, *Iterative Solution of Nonlinear Equations in Several Variables* (Academic Press, New York, 1970).
19. G. NICOLIS AND I. PRIGOGINE, *Self-organization in Nonequilibrium Systems* (Wiley, New York, 1977).
20. I. SCHREIBER, M. KUBIČEK AND M. MAREK, "On Coupled cells," in *New Approaches to Nonlinear Problems in Dynamics*, edited by P. J. Holmes (SIAM, Philadelphia, 1980), p. 496.
21. E. N. LORENZ, *J. Atmos. Sci.* **20** (1963), 130.
22. T. SHIMIZU AND N. MORIOKA, *Phys. Lett. A* **66** (1978), 192.
23. M. HOLODNIOK, M. KUBIČEK AND M. MAREK, Stable and Unstable Periodic Solutions in the Lorenz Model, Preprint, Institut für Mathematik, TUM-M8217, München, 1982.
24. K. TOMITA AND T. KAI, *Phys. Lett. A* **66** (1978), 91.
25. K. TOMITA, *Phys. Rep.* **86** (1982), 113.
26. I. SCHREIBER, M. HOLODNIOK, M. KUBIČEK, AND M. MAREK, *J. Stat. Phys.* **43** (1986), 489.
27. M. HOLODNIOK AND M. KUBIČEK, Computation of Period Doubling Bifurcation Points in Ordinary Differential Equations, preprint, Institut für Mathematik, TUM-M8406, München, 1984.
28. Z. SOMR, MS. thesis, Faculty of Math. and Physics and Prague Institute of Chem. Technology, Prague, 1984. [Czech]
29. M. KUBIČEK, M. HOLODNIOK, AND A. KLIČ, in preparation.